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Tensor Products in Quantum Functional Analysis: the Non-Matricial Approach

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Abstract. As is known, there exists an alternative, "non-matricial" way to present basic notions and results of quantum functional analysis (= operator space theory). This approach is based on considering, instead of matrix spaces, a single space, consisting, roughly speaking, of vectors from the initial linear space equipped with coefficients taken from some good operator algebra. It seems that so far there was no systematical exposition of the theory in the framework of the non-matricial approach. We believe, however, that in a number of topics the non-matricial approach gives a more elegant and transparent theory.

In this paper we introduce, using only the non-matricial language, both quantum versions of the classical (Grothendieck) projective tensor product of normed spaces. These versions correspond to the "matricial" Haagerup and operator-projective tensor products. We define them in terms of the universal property with respect to some classes of bilinear operators, corresponding to "matricial" multiplicatively bounded and completely bounded, and then produce their explicit constructions. Among the relevant results, we shall show that both tensor products are actually quotient spaces of some "genuine" projective tensor products. Moreover, the Haagerup tensor product is itself a "genuine" projective tensor product, however not of just normed spaces but of some normed modules.

This paper deals with some questions of quantum, or quantized functional analysis (cf. the memorable lecture of Effros [**1**]). Here we present some basic notions and results concerning tensor products of quantum spaces ("spaces endowed with operator space structure"). The specific feature of our exposition is that we systematically use what can be called non-matricial or non-coordinate approach to quantum functional analysis.

This means, to speak informally, the following thing. Usually a quantum norm (=operator space structure) on a given linear space is introduced by simultaneous consideration of matrices of all sizes with entries in this space. There is, however another way. Instead of these matrix spaces, one considers a single space, consisting,

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roughly speaking, of vectors from the given space equipped with coefficients taken from some good operator algebra. Such a replacing of scalars by operators, this time in the capacity of coefficients to our vectors, is, of course, well in line in the general spirit of modern "quantum", or non-commutative, mathematics.

The very fact that both approaches, matricial (coordinate) and operator (non-coordinate), give essentially equivalent results, is known. It is clearly indicated in the book of Pisier [**2**] who demonstrates the virtues of the non-matricial approach in a number of important questions (see, e.g., *idem*, p. 40). Besides, this was well realized by Barry Johnson, as one can judge from his unpublished notes. The same fact is demonstrated in the form of theorems on the equivalence of various categories [**3, 4**]; cf. also results on representations of bimodules over operator algebras [**5, 6**].

But what about concrete forms acquired by the main notions of the theory exclusively in the framework of the non-coordinate approach, without an appeal to matrix spaces? Do we obtain some new insight on the subject? It seems that there was no systematical exposition of quantum functional analysis from the indicated point of view. As to already existing monographs on the subject, even in [**2**] the matricial approach considerably prevails, and it is the only approach taken in [**7, 8, 9**].

The choice between the two approaches is, of course, the matter of taste. (One prefers to work with tensor products of linear operators, and another one with Kronecker products of matrices). However, we believe that there are some topics where the non-coordinate approach gives more elegant and transparent theory. Especially this concerns the notions where the matricial presentation inevitably creates the whole parade of indices and multi-indices, and first of all that of tensor product. (The same, as it seems to us, could be said about the duality theory, but we do not touch this topic here).

The principal aim of this paper is to introduce, using the non-coordinate language, both quantum versions of the classical (Grothendieck) projective tensor product of normed spaces. These versions are usually called Haagerup and operator-projective tensor products. We define them in terms of the universal property with respect to some classes of bilinear operators, corresponding to "matricial" multiplicatively bounded and completely bounded bilinear operators, and then produce their explicit constructions. Among the relevant results, we shall show that both tensor products are actually quotient spaces of some "genuine" projective tensor products. Moreover, the Haagerup tensor product is itself a "genuine" projective tensor product, however not of just normed spaces but of some normed modules.[1]

¿From the huge number of substantial examples of quantum spaces we use here only one pair of very illustrative twin spaces, the so-called column and row Hilbertians. Their habits and, in particular, their behaviour as tensor factors are well known in the matricial exposition; see, e.g., [**7**, Section 9.3]. We just want to show that these things, providing the adornment of general results, well fit in the non-coordinate presentation.

[1]The author gave talks about these results at the conferences in Athens (June 2005) and Bordeaux (July 2005). After the second talk Dr. C. -K. Ng, who was present, kindly informed the author that the results are essentially known to him, and that the similar things are contained in his yet unpublished preprints.

1. Preparing the stage

Throughout the paper, the terms operator and bioperator mean always respectively linear operator and bilinear operator; the terms functional and bifunctional have the similar meaning. If E and F are normed spaces, then $\mathcal{B}(E,F)$ and $\mathcal{K}(E,F)$ denote, as usual, the space of all bounded, respectively compact, operators between these spaces, $\mathcal{B}(E)$ means $\mathcal{B}(E,E)$ and $\mathcal{K}(E)$ means $\mathcal{K}(E,E)$. The identity operator on E is denoted by $\mathbf{1}_E$ or, if it is safe, just by $\mathbf{1}$.

The term *operator space* will be used for an arbitrary (not necessarily closed) subspace in $\mathcal{B}(H,K)$ for some *Hilbert spaces* H and K. So far we do not equip operator spaces with any additional structure save induced (= operator) norm. The symbol $\dot{\otimes}$ denotes, according to the sense, one of the following three things. Namely, $H \dot{\otimes} K$ is the Hilbert tensor product of respective Hilbert spaces whereas $a \dot{\otimes} b$ is the Hilbert tensor product of respective bounded operators acting between Hilbert spaces (see, e.g.,[**10**, Ch.2, §8]). Finally, if $E \subseteq \mathcal{B}(H_1,K_1)$ and $F \subseteq \mathcal{B}(H_2,K_2)$ are operator spaces, then $E \dot{\otimes} F$ is the so-called (non-completed) *spatial tensor product* of E and F, that is the operator space $span\{a \dot{\otimes} b; a \in E, b \in F\} \subseteq \mathcal{B}(H_1 \dot{\otimes} H_2, K_1 \dot{\otimes} K_2)$.

If H is a Hilbert space and $\xi,\eta \in H$, we denote by $\xi \bigcirc \eta : H \to H$ the rank one operator taking ζ to $\langle \zeta,\eta\rangle\xi$. Let us distinguish the obvious equalities

$$(\xi \bigcirc \eta)(\xi' \bigcirc \eta') = \langle \xi',\eta\rangle(\xi \bigcirc \eta'), a(\xi \bigcirc \eta) = a(\xi) \bigcirc \eta \quad \text{and} \quad \|\xi \bigcirc \eta\| = \|\xi\|\|\eta\|, \quad (1)$$

where $\xi,\xi'\eta,\eta' \in H, a \in \mathcal{B}(H)$.

For a short time, consider an arbitrary unital $\mathcal{B}$-bimodule X. Take $u \in X$. We call every projection (= self-adjoint idempotent) $P \in \mathcal{B}$ a *left* (respectively, *right*) *support* of the element u, if $P \cdot u = u$ (respectively, $u \cdot P = u$. If we have both equalities, we speak about (just) a *support* of u.

Let $\|\cdot\|$ be a semi-norm on X. We say that it satisfies the *first axiom of Ruan* (briefly, (RI)) if, for every $a \in \mathcal{B}$ and $u \in X$ we have $\|a\cdot u\|, \|u \cdot a\| \leq \|a\|\|u\|$. (In the usual language of the theory of Banach algebras this means exactly that X is a contractive, or linked semi-normed $\mathcal{B}$-bimodule). We distinguish the obvious

Proposition 1. *Suppose that the semi-norm on X satisfies (RI). Then for every $u \in X$, isometric operator $a \in \mathcal{B}$ and every coisometric operator $b \in \mathcal{B}$ we have $\|a\cdot u\| = \|u\cdot b\| = \|u\|$.*

Further, we say that a semi-norm $\|\cdot\|$ in X satisfies the *second axiom of Ruan* (briefly, (RII)) if, whenever u (respectively, v) in X has a support P (respectively, Q), and these projections are orthogonal (i.e. $PQ = 0$), we have $\|u+v\| = \max\{\|u\|,\|v\|\}$. (We choose both axioms as "non-coordinate" versions of the known Ruan axioms for matrix-norms (cf., e.g., [**7**, p.20] or [**8**, p.180-181]).

Proposition 2. *Let a semi-norm $\|\cdot\|$ on X satisfy both axioms of Ruan, and elements $u_k \in X; k = 1,...,n$ have pairwise orthogonal left supports or pairwise orthogonal right supports. Then $\|u_1 + ... + u_n\| \leq (\|u_1\|^2 + ... + \|u_n\|^2)^{\frac{1}{2}}$.*

$\triangleleft$ Obviously, we can assume that $\|u_k\| \neq 0$ for all k. Let us concentrate on the case of left supports, say P_k. At first we shall show that the assertion is true if we assume, in addition, that our supports, as operators on L, have infinite-dimensional images.

The latter condition enables us to choose isometric operators $S_k \in \mathcal{B}; k = 1, ..., n$ such that $S_k S_k^* = P_k$. Consider, for every k, $v_k := \frac{1}{\|u_k\|} u_k \cdot S_k^* \in \mathcal{K}E$. It is easy to see that P_k is a (two-sided) support of v_k. Moreover, by Proposition 1, we have $\|v_k\| = 1$ for all k. Consequently, the axiom (RII), extended by obvious way to the case of n elements, gives $\|v_1 + ... + v_n\| = 1$. Now put $a_k := \|u_k\| S_k \in \mathcal{B}$. Then we have

$$(v_1 + ... + v_n) \cdot (a_1 + ... + a_n) = \sum_{k,l=1}^{n} [\frac{1}{\|u_k\|} u_k \cdot S_k^*] \cdot \|u_l\| S_l =$$

$$\sum_{k,l=1}^{n} [\frac{1}{\|u_k\|} \|u_l\| u_k] \cdot S_k^* S_l = u_1 + ... + u_n.$$

Therefore, again by (RI), we have $\|u_1 + ... + u_n\| \leq \|a_1 + ... + a_n\| \|v_1 + ... + v_n\| = \|a_1 + ... + a_n\|$. But routine calculations, using the C^*-identity, show that the norm of the operator $a_1 + ... + a_n$ is exactly $(\|u_1\|^2 + ... + \|u_n\|^2)^{\frac{1}{2}}$. The desired estimation follows.

In the case of arbitrary left supports P_k we use the following device. Let nL be a Hilbert sum of n copies L_k of L, and $Q_k : nL \to nL$ a projection onto L_k. Obviously, there exists an isometric operator $R : L \to nL$, mapping, for every k, $Im(P_k)$ into L_k. Choose an arbitrary isometric isomorphism $U : nL \to L$ and note that the operator $UR \in \mathcal{B}$ is isometric.

Now put $u'_k := UR \cdot u_k; k = 1, ..., n$. Of course, operators $P'_k := UQ_kU^* \in \mathcal{B}; k = 1, ..., n$ are pairwise orthogonal projections with infinite-dimensional images. Moreover, we have

$$P'_k \cdot u'_k = UQ_kU^* \cdot (UR \cdot u_k) = (UQ_kU^*UR) \cdot u_k = (UQ_kR) \cdot u_k.$$

But the choice of R obviously implies, for all k, that $Q_kRP_k = RP_k$. Therefore

$$P'_k \cdot u'_k = (UQ_kR) \cdot (P_k \cdot u_k) = (UQ_kRP_k) \cdot u_k = UR \cdot (P_k \cdot u_k) = UR \cdot u_k = u'_k.$$

Thus we find ourselves in the situation, where the desired inequality, however with u'_k instead of u_k, is already established. But Proposition 1 immediately gives $\|u'_k\| = \|u_k\|$ and $\|\sum_{k=1}^n u'_k\| = \|\sum_{k=1}^n u_k\|$. $\triangleright$

We proved the part of the assertion, concerning left supports. The analogous argument provides the part, concerning right supports.

Remark. This proposition, at least for normed bimodules, could be obtained as a corollary to some deep results about representations of some bimodules over C^*-algebras in terms of the so-called operator convexity (see., e.g., [**5, 6**]). But, since our aims are quite different, we do not need any of such a strong medicine here.

Now we leave general $\mathcal{B}$-bimodules. Choose an arbitrary separable infinite-dimensional Hilbert space, denote it, say, by L and fix it throughout the whole scope of this paper. The operator algebras $\mathcal{B}(L)$ and $\mathcal{B}(L)$ we denote, for brevity, by $\mathcal{B}$ and $\mathcal{K}$. Instead of $\mathbf{1}_L$ we shall always write just $\mathbf{1}$.

Let E be a linear space. Denote, again for brevity, the algebraic tensor product $\mathcal{K} \otimes E$ by $\mathcal{K}E$ and call this space the *amplification* of E. This is, to speak informally, "the space of formal linear combinations of vectors from E with operator coefficients from $\mathcal{K}$". Accordingly, we denote an elementary tensor $a \otimes x; a \in \mathcal{K}, x \in E$ just by ax. Observe that the space $\mathcal{K}E$ is a $\mathcal{B}$-bimodule with respect to the outer

multiplications, well defined by the equalities $a \cdot bx = (ab)x$ and $bx \cdot a = (ba)x; a \in \mathcal{B}, b \in \mathcal{K}, x \in E$.

In the following two propositions we consider a $\mathcal{B}$-bimodule of the form $\mathcal{K}E$ and we suppose that it is equipped with a semi-norm $\| \cdot \|$ satisfying (RI).

Proposition 3. *Assume that* $\lim_{n\to\infty} a_n = 0$ *for some* $a_n \in \mathcal{K}$. *Then, for every* $x \in E$, *we have* $\lim_{n\to\infty} a_n x = 0$.

$\triangleleft$ As is well known, there exist $b_n, c \in \mathcal{K}$ such that $a_n = b_n c; n = 1, 2, \ldots$ and $\lim_{n\to\infty} b_n = 0$ (see, e.g., [**11**, §11, Cor.12]). Hence, by (RI), $\|a_n x\| = \|b_n \cdot (cx)\| \leq \|b_n\| \|cx\|$. The rest is clear. $\triangleright$

Proposition 4. *Assume that, for some projection* $p \in \mathcal{K}$ *of rank 1 the restriction of the given semi-norm on* $\mathcal{K}E$ *to the subspace* $\{px; x \in E\}$ *is a norm. Then the given semi-norm is itself a norm.*

$\triangleleft$ Take a non-zero element $u \in \mathcal{K}E$. Then, as is well known (see, e.g., [**10**, Proposition 2.7.1]), it can be represented as $\sum_{k=1}^n a_k x_k$, where $a_k; k = 1, \ldots, n$ is a linearly independent system of compact operators, and $x_1 \neq 0$. Consider the system of all vector functionals on $\mathcal{K}$, that is of those acting as $a \mapsto \langle a\xi, \eta \rangle; \xi, \eta \in H$. Of course, the system of vector functionals is sufficient, i.e. for every $a \neq 0$ there exists a vector functional with a non-zero value on a. According to the known property of linear span of sufficient systems (see, e.g., [**10**, Proposition 4.2.3]), there exist $\xi_l, \eta_l \in H; l = 1, \ldots, m$ such that $\sum_{l=1}^m \langle a_k \xi_l, \eta_l \rangle$ is 1 when $k = 1$ and is 0 otherwise.
Now take a normed vector $e \in Im(p)$; obviously, $p = e \bigcirc e$. Consider in $\mathcal{K}E$ the element $v := \sum_{l=1}^m (e \bigcirc \eta_l) \cdot u \cdot (\xi_l \bigcirc e)$ and recall the equalities (1). We see that

$$v = \sum_{l=1}^m (e \bigcirc \eta_l) \cdot (\sum_{k=1}^n a_k x_k) \cdot (\xi_l \bigcirc e) = \sum_{k,l} [(e \bigcirc \eta_l) a_k (\xi_l \bigcirc e)] x_k =$$

$$\sum_{k,l} [\langle a_k \xi_l, \eta_l \rangle p] x_k = [\sum_{l=1}^m \langle a_1 \xi_l, \eta_l \rangle p] x_1 = p x_1.$$

Therefore, by our assumption, $\|v\| \neq 0$. At the same time the axiom (RI) and the triangle inequality for semi-norms give

$$\|v\| \leq \sum_{l=1}^m \|(e \bigcirc \eta_l) \cdot u \cdot (\xi_l \bigcirc e)\| \leq \sum_{l=1}^m \|e \bigcirc \eta_l\| \|u\| \|\xi_l \bigcirc e\|.$$

This, of course, implies $\|u\| > 0$. $\triangleright$

2. Quantum spaces and completely bounded operators

The main concepts of quantum functional analysis, in its non-coordinate presentation, are those given in the following definition and in still more important Definition 2.

Definition 1. A *quantum* norm on a linear space E is an arbitrary norm on the $\mathcal{B}$-bimodule $\mathcal{K}E$, satisfying both of Ruan axioms. A *quantum normed space* or just a *quantum space* is a linear space, equipped with a quantum norm.

(We emphasize that a quantum norm on E is a (usual) norm not on E itself, but on the "larger" space $\mathcal{K}E$).

A quantum normed space becomes a "classical" normed space, if, for $x \in E$, we put $\|x\| := \|px\|$, where p is an arbitrary projection of rank 1 on L. It easily follows

from Proposition 1 that this is indeed a norm, not depending on a choice of p. (In fact, this norm does not change if we replace p by an arbitrary $a \in \mathcal{K}; \|a\| = 1$. But we do not need this fact here). The resulting normed space, often denoted by $\square E$, is called the *underlying normed space* of the quantum space E. As to the initial quantum space, we call it a *quantization* of its underlying space $\square E$, and we call its quantum norm a *quantization* of the "usual" norm on $\square E$. We shall see that the same, up to isometric isomorphism, normed space can have a lot of profoundly different quantizations. However the simplest normed space, the complex plane $\mathbb{C}$, has a unique quantization. Namely, it easily follows from axioms of Ruan that the operator norm on $\mathcal{K}\mathbb{C} = \mathcal{K}$ is the only quantization of the norm on $\mathbb{C}$.

Let F be a linear subspace of a quantum space E. Then F becomes itself a quantum space with respect to the norm on $\mathcal{K}F$, well defined by $\|u\| := \|\mathbf{1}_{\mathcal{K}} \otimes i\|$, where $i : F \to E$ is the natural embedding. In this situation we say that F is a *quantum subspace* of E.

We turn to a general construction providing the principal class of quantum spaces.

Assume that a linear space E is given together with an injective operator $I : E \to \mathcal{B}(H, K)$ for some Hilbert spaces H and K. Clearly, E becomes a normed space with respect to the induced norm, and I becomes an isometric operator which provides the identification of this normed space with the operator space $I(E)$. But more can be said in this situation. Consider the operator $J : \mathcal{K}E \to \mathcal{B}(L\dot{\otimes}H, L\dot{\otimes}K)$, associated with the bioperator $\mathcal{K} \times E \to \mathcal{B}(L \dot{\otimes} H, L \dot{\otimes} K) : (a, x) \mapsto a \dot{\otimes} I(x)$. It is well known (and easy to check) that J is injective. Therefore we can endow $\mathcal{K}E$ with the respective induced norm, thus identifying $\mathcal{K}E$ with the operator space $J(\mathcal{K}E) = \mathcal{K} \dot{\otimes} E$. It is easy to verify that this norm on $\mathcal{K}E$ is a quantum norm on E, moreover a quantization of the usual norm on the latter space. This quantum norm on E, as well as respective quantum space, are called *concrete* quantum norm or, respectively, quantum space (*associated to the injective operator* I, if we want to be precise).

Remark. As a matter of fact, every ("abstract") quantum norm on a linear space is a concrete quantum norm (associated to some I). This is the famous Ruan Theorem (see, e.g., [**7**, p.33]), or, more accurately, its non-coordinate version. However, we do not need this deep theorem here.

If a linear space E is already presented as an operator space, we always take as I the respective natural embedding and call the resulting quantum norm and quantum space *standard*. The term "standard quantization" of an operator space or of its norm has the similar meaning. We see that in the indicated case we just identify $\mathcal{K}E$ with the operator space $\mathcal{K} \dot{\otimes} E$.

Let us distinguish two important particular cases of the concrete quantization that will provide instructive illustrations to our future quantum tensor products.

Take a Hilbert space, say, H. Consider linear (and isometric) isomorphism $I_c : H \to \mathcal{B}(\mathbb{C}, H)$, taking x to the operator $1 \mapsto x$, as well as another linear (and isometric) isomorphism $I_r : H \to \mathcal{B}(\overline{H}, \mathbb{C})$, taking x to the functional $y \mapsto \langle x, y \rangle$. (Here and thereafter $\overline{H}$ denote the complex-conjugate space of H). Endow H with two concrete quantum norms associated respectively with I_c and I_r, and denote the

resulting quantum spaces by H_c and H_r. Obviously the underlying normed space of both H_c and H_r is H with its original norm.

The quantum space H_c (respectively, H_r) is called the *column* (respectively, *raw*) quantization of the Hilbert space H, or, if H is fixed, the *column* (respectively, *raw*) Hilbertian.

The following observation considerably facilitates the work with these Hilbertians.

Proposition 5. *Let H be a Hilbert space, E is a linear space. Then every element w in $E \otimes H$ has the form $\sum_{k=1}^n x_k \otimes e_k$, where e_k is an orthonormal system in H, and $x_k \in E$. Moreover, if E is an operator space, and H is identified with the operator space $I_c(H)$, respectively, $I_r(H)$, then w, being considered in the operator space $E \dot{\otimes} H$, has the norm*

$$\|w\| = \| \sum_{k=1}^n x_k^* x_k \|^{\frac{1}{2}}, \quad \text{respectively,} \quad \|w\| = \| \sum_{k=1}^n x_k^* x_k \|^{\frac{1}{2}}.$$

$\triangleleft$ To obtain the indicated representation, we take an arbitrary representation, say, $\sum_{k=1}^m y_k \otimes \xi_k$, of w, take an orthonormal basis e_k in $span\{\xi_1, .., \xi_m\} \subset H$ and use the bilinearity of the operation "$\otimes$".

To compute $\|w\|$ in the "column" case, we note the following. Our e_k, now operators in $\mathcal{B}(\mathbb{C}, H)$, satisfy $e_k^* e_l = \delta_l^k \mathbf{1}_{\mathbb{C}}$. (Here and thereafter δ is the symbol of Kronecker). Combining this with the operator C^*-identity $\|w\| = \|w^* w\|^{\frac{1}{2}}$, we easily get the desired expression.

Similar argument works in the "raw" case as well. The only modification is that now we have $e_k e_l^* = \delta_l^k \mathbf{1}_{\mathbb{C}}$ and use the C^*-identity in the form $\|w\| = (\|ww^*\|)^{\frac{1}{2}}$. $\triangleright$

Consider an important particular case of the obtained equalities. As usual, for a given partial isometry, say S, on some Hilbert space, the operator S^*S will be called its initial, and SS^* its final projection.

Now let $q_k \in \mathcal{K}; k = 1, ..., n$ be arbitrary (of course, finite-dimensional) partial isometries in $\mathcal{K}$ with the same initial projection P and with pairwise orthogonal final projections. Besides, let $e_1, ..., e_n$ be an orthonormal system in H. Put

$$\omega := \sum_{k=1}^n q_k^* e_k \in \mathcal{K}H \quad \text{and} \quad \varpi := \sum_{k=1}^n q_k e_k \in \mathcal{K}H. \tag{2}$$

Proposition 6. $\widehat{\text{I}}$f we consider the quantum space H_c, then $\|\omega\| = 1$ whereas $\|\varpi\| = \sqrt{n}$. At the same time, if we consider H_r, then $\|\omega\| = \sqrt{n}$ whereas $\|\varpi\| = 1$.

$\triangleleft$ Take $\mathcal{K}$ as E and do what is prescribed by Proposition 5. Then we see that in the "column" case $\|\omega\|^2$ is the norm of the operator $\sum_{k=1}^n q_k q_k^*$ whereas $\|\varpi\|^2$ is that of $\sum_{k=1}^n q_k^* q_k = nP$. The assertion in the column case immediately follows. A similar argument establishes the "raw" case. $\triangleright$

After the introducing, by Definition 1, a certain additional structure on linear spaces, we naturally proceed to the discussion of maps, reacting in an appropriate way to this structure.

Let $\varphi : E \to F$ be an operator between linear spaces. The operator $\mathbf{1}_{\mathcal{K}} \otimes \varphi : \mathcal{K}E \to \mathcal{K}F$, denoted for brevity by φ_∞, is called the *amplification* of φ. Note that φ_∞ is a morphism of $\mathcal{B}$-bimodules (cf. above).

Definition 2. Let E and F be quantum spaces. The operator $\varphi : E \to F$ is called *completely bounded* if its amplification φ_∞ is a bounded operator (with respect to the relevant quantum norms). The operator norm of φ_∞ is called *completely bounded norm* of φ and is denoted by $\|\varphi\|_{cb}$. Further, the operator φ is called *completely contractive* if φ_∞ is contractive (i.e. $\|\varphi\|_{cb} \leq 1$), *completely isometric* if φ_∞ is isometric and *completely isometric isomorphism* if φ_∞ is an isometric isomorphism.

If an operator $\varphi : E \to F$ between quantum spaces is bounded as an operator between the respective underlying normed spaces, we say that it is (just) bounded. Taking an arbitrary rank 1 projection p and passing from φ_∞ to its birestriction, which acts between $\{px; x \in E\}$ and $\{py; y \in F\}$, we see that *every completely bounded operator is bounded*, and $\|\varphi\| \leq \|\varphi\|_{cb}$.

In a number of important situations the converse is also true. We need here only one result of that kind (cf. the "matricial" Corollary 2.2.3 in [**7**]).

Proposition 7. *Let $f : E \to \mathbb{C}$ be a bounded functional on a quantum space. Then it is (automatically) completely bounded, and $\|f\|_{cb} = \|f\|$.*

$\triangleleft$ Consider $f_\infty : \mathcal{K}E \to \mathcal{K}\mathbb{C} = \mathcal{K}$ and take $u \in \mathcal{K}E$. By virtue of properties of the operator norm, we have

$$\|f_\infty(u)\| = \sup\{|\langle f_\infty(u)\xi, \eta\rangle|; \xi, \eta \in L, \|\xi\|, \|\eta\| \leq 1\}.$$

Fix an arbitrary normed vector $e \in L$ and take the projection $p = e \bigcirc e$ onto its linear span. Using the first and the second of the equalities (1) and then the morphism property of f_∞ (see above), we have

$$\langle [f_\infty(u)](\xi), \eta\rangle p = \langle [f_\infty(u)](\xi), \eta\rangle (e \bigcirc e) =$$

$$(e \bigcirc \eta)([f_\infty(u)](\xi) \bigcirc e) = (e \bigcirc \eta) f_\infty(u)(\xi \bigcirc e) = f_\infty[(e \bigcirc \eta) \cdot u \cdot (\xi \bigcirc e)].$$

Therefore $|\langle [f_\infty(u)](\xi), \eta\rangle| = \|f_\infty[(e \bigcirc \eta) \cdot u \cdot (\xi \bigcirc e)]\|$.

Now observe that $(e \bigcirc \eta) \cdot u \cdot (\xi \bigcirc e)$ is an elementary tensor of the form $px_{\xi,\eta}$ for some $x_{\xi,\eta} \in E$. (Obviously, it is the case when u is an elementary tensor, and hence it is true for all u). Besides, it follows from (RI) and from the third equality in (1) that $\|x_{\xi,\eta}\| = \|x_{\xi,\eta} p\| \leq \|e \bigcirc \eta\| \|u\| \|\xi \bigcirc e\| \leq \|u\|$ wherever $\|\xi\|, \|\eta\| \leq 1$. Hence for the same ξ, η we have

$$|\langle [f_\infty(u)](\xi), \eta\rangle| = \|f_\infty(px_{\xi,\eta})\| = \|f(x_{\xi,\eta})p\| = |f(x_{\xi,\eta})| \leq \|f\| \|x_{\xi,\eta}\| \leq \|f\| \|u\|.$$

Taking the respective supremum, we see that $\|f_\infty\| \leq \|f\|$. The rest is clear. $\triangleright$

However, the "usual" boundedness, generally speaking, does not imply the complete boundedness, and this is a fundamental observation of the whole theory. Probably, the simplest and most illuminating counter-example is provided by the identity operator $\mathbf{1} : H_c \to H_r$ where H is an infinite-dimensional Hilbert space. Indeed, by virtue of Proposition 6, for every n one can find an element in $\mathcal{K}H$ such that the amplification $\mathbf{1}_\infty : \mathcal{K}H_r \to \mathcal{K}H_c$ increases its norm exactly in $\sqrt{n}$ times. Thus $\mathbf{1}_\infty$ is not bounded and hence the original operator, being "on the level of underlying normed spaces" even isometric, is not completely bounded. The same, with obvious modifications, can be said about the operator $\mathbf{1} : H_r \to H_c$.

3. Completely bounded bilinear operators

As is known, there is a universal consent in the classical functional analysis concerning what to call bounded bioperator between normed spaces. As to quantum functional analysis, the experience of last 15 years has shown that there exist at least two versions of the notion of completely bounded bioperator, each with its own advantages. We begin with the earlier version, discovered (in the "matricial" presentation) by Christensen and Sinclair [**12**], 1987.

Let $\mathcal{R} : E \times F \to G$ be a bioperator, connecting three linear spaces. Consider the bioperator $\mathcal{R}_s : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}G$, associated with the 4-linear operator $\mathcal{K} \times E \times \mathcal{K} \times F \to \mathcal{K}G : (a, x, b, y) \mapsto ab\mathcal{R}(x, y)$. (Otherwise, $\mathcal{R}_s$ is well-defined by taking a pair (ax, by) to $ab\mathcal{R}(x, y)$). This bioperator is called the *strong amplification* of $\mathcal{R}$.

Definition 3. Let E, F and G be quantum spaces. A bioperator $\mathcal{R} : E\times F \to G$ is called *strongly completely bounded*[2] if its strong amplification $\mathcal{R}_s$ is a bounded bioperator (with respect to the relevant quantum norms). The bioperator norm of $\mathcal{R}_s$ is called *strong completely bounded norm* of $\mathcal{R}$ and is denoted by $\|\mathcal{R}\|_{scb}$. Further, the bioperator $\mathcal{R}$ is called *strongly completely contractive* if $\mathcal{R}_s$ is contractive (i.e. $\|\mathcal{R}\|_{scb} \leq 1$).

In order to introduce another version of complete boundedness for bioperators, we need some preparation. We would like to have an operation that imitates the tensor multiplication of operators on our fixed Hilbert space L, but does not lead out of this space.

By virtue of Fischer-Riesz Theorem, there exists a unitary isomorphism $\iota : L \to L \dot{\otimes} L$. Take one and fix it throughout this paper. (It does not matter which one we choose). Our ι gives rise to the isometric *-isomorphism $\varkappa := \mathcal{B}(L \dot{\otimes} L) \to \mathcal{B} : a \mapsto \iota^* a \iota$.

Let us use, for the operator $\varkappa(a \dot{\otimes} b) \in \mathcal{B}; a, b \in \mathcal{B}$, the brief notation $a\diamondsuit b$. Obviously, we have the identities

$$(a\diamondsuit b)(c\diamondsuit d) = ac\diamondsuit bd, (a\diamondsuit b)^* = a^*\diamondsuit b^* \quad \text{and} \quad \|a\diamondsuit b\| = \|a\|\|b\|. \tag{3}$$

Besides, $a, b \in \mathcal{K}$ implies $a\diamondsuit b \in \mathcal{K}$.

Now let $\mathcal{R}$ be as above. Consider the bioperator $\mathcal{R}_w : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}G$, associated with the 4-linear operator $\mathcal{K} \times E \times \mathcal{K} \times F \to \mathcal{K}G : (a, x, b, y) \mapsto (a\diamondsuit b)\mathcal{R}(x, y)$ (and well-defined by taking (ax, by) to $(a\diamondsuit b)\mathcal{R}(x, y)$). This bioperator is called the *weak amplification* of $\mathcal{R}$.

Definition 4. Let E, F and G be quantum spaces. A bioperator $\mathcal{R} : E\times F \to G$ is called *weakly completely bounded* if its weak amplification is a bounded bioperator. The bioperator norm of $\mathcal{R}_w$ is called *weak completely bounded norm* of $\mathcal{R}$ and is denoted by $\|\mathcal{R}\|_{wcb}$. The bioperator $\mathcal{R}$ is called *weakly completely contractive* if $\mathcal{R}_w$ is contractive.

Now let us widen the field of action of the operation "diamond". Namely, for a linear space E and $a \in \mathcal{K}$ we consider the operators ${}_a\diamondsuit, \diamondsuit_a : \mathcal{K}E \to \mathcal{K}E$, associated

[2] In the pioneering paper [**12**] and in a lot of other papers and books, up to the present time, such a bioperator (or, more precisely, its matricial version) is called just completely bounded. However, in some other books and papers, notably in the influential textbook of Effros and Ruan [**7**], it is called multiplicatively bounded whereas the term "completely bounded" is used for the "matricial prototype" of what we call here weakly completely bounded bioperator.

with the bioperators $\mathcal{K} \times E \to \mathcal{K}E$ taking (b, x) respectively to $(a\diamondsuit b)x$ and $(b\diamondsuit a)x$. Then for $a \in \mathcal{K}$ and $u \in \mathcal{K}E$ we put $a\diamondsuit u :=_a \diamondsuit(u)$ and $u\diamondsuit a := \diamondsuit_a(u)$. Obviously, both new "diamond multiplications" are uniquely determined by their bilinearity and the equations

$$a\diamondsuit bx = (a\diamondsuit b)x, \quad \text{respectively} \quad bx\diamondsuit a = (b\diamondsuit a)x; \quad a, b \in \mathcal{K}, x \in E.$$

Mention the useful formulae

$$(a\diamondsuit b) \cdot (c\diamondsuit u) = ac\diamondsuit(b \cdot u), \quad (a\diamondsuit u) \cdot (b\diamondsuit c) = ab\diamondsuit(u \cdot c),$$
$$(a\diamondsuit b) \cdot (u\diamondsuit c) = (a \cdot u)\diamondsuit bc \quad \text{and} \quad (u\diamondsuit a) \cdot (b\diamondsuit c) = (u \cdot b)\diamondsuit ac, \tag{4}$$

where $u \in \mathcal{K}E$, and other letters denote operators in $\mathcal{K}$ or, if it is sensible, in $\mathcal{B}$. (With the help of the first equality in (3), one can easily check them for elementary tensors and then use the bilinearity).

Proposition 8. *Let E be a quantum space and $P \in \mathcal{K}$ a projection of finite rank. Then, for every $u \in \mathcal{K}E$, we have $\|P\diamondsuit u\| = \|u\diamondsuit P\| = \|u\|$.*

$\triangleleft$ Let us begin with a projection of rank one, say, p. Fix a normed vector, say e, in its image and consider the isometric operator $\rho : L \to L \dot{\otimes} L : \xi \mapsto e \otimes \xi$. Since ρ^* is uniquely determined by the taking $e \otimes \xi$ to ξ and $e' \otimes \xi$ to 0 for all $e' : e' \perp e$, we easily see that $\rho a \rho^* = p \dot{\otimes} a$ for all $a \in \mathcal{B}$. Therefore if we introduce the isometric operator $S_p := \iota^* \rho \in \mathcal{B}$, we have

$$S_p a S_p^* = \iota^* \rho a \rho^* \iota = \iota^*(p \dot{\otimes} a)\iota = \varkappa(p \dot{\otimes} a) = p\diamondsuit a.$$

Consequently, we have $p\diamondsuit u = S_p \cdot u \cdot S_p^*$ for all elementary tensors in $\mathcal{K}E$ and hence, by bilinearity, for all $u \in \mathcal{K}E$. Proposition 1 immediately implies $\|p\diamondsuit u\| = \|u\|$.

Now let P be a projection of rank N on L. Then, for some pairwise orthogonal projections $p_1, ..., p_N$ of rank one, we have $P = \sum_{k=1}^N p_k$. Take $u \in \mathcal{K}E$. Then $P\diamondsuit u = \sum_{k=1}^N p_k\diamondsuit u$, and elements $p_k\diamondsuit u$ have pairwise orthogonal supports, namely $p_k\diamondsuit \mathbf{1}$. Therefore the obvious extension of (RII) to the case of several elements gives $\|P\diamondsuit u\| = \max\{\|p_k\diamondsuit u\|; k = 1, ..., n\}$. Hence $\|P\diamondsuit u\| = \|u\|$. A similar argument provides $\|u\diamondsuit P\| = \|u\|$. $\triangleright$

Theorem 1. (cf. the "matricial prototype" in [**7**, p.150]) *Let $\mathcal{R} : E \times F \to G$ be a strongly completely bounded bioperator between quantum spaces. Then $\mathcal{R}$ is weakly completely bounded, and $\|\mathcal{R}\|_{wcb} \leq \|\mathcal{R}\|_{scb}$.*

$\triangleleft$ Take a projection P of finite rank on L. For elementary tensors $ax \in \mathcal{K}E$ and $by \in \mathcal{K}F$ we have

$$\mathcal{R}_w([ax] \cdot P, P \cdot [by]) = \mathcal{R}_w([aP]x, [Pb]y) = (aP\diamondsuit Pb)\mathcal{R}(x, y) =$$
$$(a\diamondsuit P)(P\diamondsuit b)\mathcal{R}(x, y) = \mathcal{R}_s([a\diamondsuit P]x, [P\diamondsuit b]y) = \mathcal{R}_s(ax\diamondsuit P, P\diamondsuit by).$$

Therefore, by bilinearity, we have $\mathcal{R}_w(u \cdot P, P \cdot v) = \mathcal{R}_s(u\diamondsuit P, P\diamondsuit v)$ for all $u \in \mathcal{K}E$ and $v \in \mathcal{K}F$. Hence $\|\mathcal{R}_w(u \cdot P, P \cdot v)\| \leq \|\mathcal{R}_s\|\|u\diamondsuit P\|\|P\diamondsuit v\|$ and, taking into account the previous proposition, we have

$$\|\mathcal{R}_w(u \cdot P, P \cdot v)\| \leq \|\mathcal{R}_s\|\|u\|\|v\|. \tag{5}$$

Now take a sequence P_n of finite-dimensional projections in L, providing an approximate identity in $\mathcal{K}$. Obviously, we have $\lim_{n\to\infty} aP_n\diamondsuit P_n b = a\diamondsuit b$ for all $a, b \in \mathcal{K}$. Therefore Proposition 3 easily implies that $\lim_{n\to\infty} \mathcal{R}_w(u \cdot P_n, P_n \cdot v) = \mathcal{R}_w(u, v)$ in $\mathcal{K}G$ for all elementary tensors $u \in \mathcal{K}E, v \in \mathcal{K}F$. Hence, by bilinearity, the same

is true for all $u \in \mathcal{K}E, v \in \mathcal{K}F$. Combining this with (5), we have $\|\mathcal{R}_w(u,v)\| \leq \|\mathcal{R}_s\|\|u\|\|v\|$. The rest is clear. $\triangleright$

Similarly to what we have seen in the case of operators, a weakly (and hence strongly) completely bounded operator $\mathcal{R}$ between concrete quantum spaces is automatically bounded as bioperator between the respective underlying spaces, and we have $\|\mathcal{R}\| \leq \|\mathcal{R}\|_{wcb}$. Indeed, if $p \in \mathcal{K}$ is a projection of rank one, the same is true for $p\diamondsuit p$. Therefore for $x \in E$ and $y \in F$ we have

$$\|\mathcal{R}(x,y)\| = \|(p\diamondsuit p)\mathcal{R}(x,y)\| = \|\mathcal{R}_w(px,py)\| \leq \|\mathcal{R}\|_{wcb}\|px\|\|py\| = \|\mathcal{R}\|_{wcb}\|x\|\|y\|,$$

and the desired fact follows.

Again, like in the case of operators, in a number of concrete situations the converse is true.

Proposition 9. *Suppose that E and F be quantum spaces, f and g are bounded functionals respectively on E and F, and $f \times g : E \times F \to \mathbb{C}$ is the bifunctional, acting as $(x,y) \mapsto f(x)g(y)$. Then $f \times g$ is strongly and hence weakly completely bounded, and $\|f \times g\|_{scb} = \|f \times g\|_{wcb} = \|f\|\|g\|$.*

$\triangleleft$ Obviously we have $\|f \times g\| = \|f\|\|g\|$ and hence $\|f\|\|g\| \leq \|(f \times g)_w\|$. Therefore, by virtue of Theorem 1, it is sufficient to show that $\|(f \times g)_s\| \leq \|f\|\|g\|$.

Taking elementary tensors and using the bilinearity, we easily see that $(f \times g)_s : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}$ acts as $(u,v) \mapsto f_\infty(u)g_\infty(v)$. ¿From this, with the help of Proposition 7, we have

$$\|(f \times g)_s(u,v)\| \leq \|f_\infty(u)\|\|g_\infty(v)\| \leq \|f\|\|g\|\|u\|\|v\|.$$

The rest is clear. $\triangleright$

The following property of weakly completely bounded bioperators, as we shall soon see, has no "strong" analogue. For a bioperator $\mathcal{R} : E \times F \to G$, acting between linear spaces, put $\mathcal{R}^{op} : F \times E \to G : (y,x) \mapsto \mathcal{R}(x,y)$.

Proposition 10. *Suppose that $\mathcal{R}$ acts between quantum spaces, and it is weakly completely bounded. Then $\mathcal{R}^{op}$ is also weakly completely bounded, and $\|\mathcal{R}^{op}\|_{wcb} = \|\mathcal{R}\|_{wcb}$.*

$\triangleleft$ Consider the flip operator $\bigtriangledown : L \dot{\otimes} L \to L \dot{\otimes} L$, well-defined by $\xi \otimes \eta \mapsto \eta \otimes \xi; \xi, \eta \in L$. It gives rise to another unitary operator, namely $\bigtriangleup := \iota^* \bigtriangledown \iota : L \to L$. Since $\bigtriangledown(a \dot{\otimes} b)\bigtriangledown = b \dot{\otimes} a; a, b \in \mathcal{B}$, we have, for the same a, b the equality $\bigtriangleup(a\diamondsuit b)\bigtriangleup = b\diamondsuit a$.

Now consider $(\mathcal{R}^{op})_w : \mathcal{K}F \times \mathcal{K}E \to \mathcal{K}G$. It easily follows from the latter equality that we have $(\mathcal{R}^{op})_w(v,u) = \bigtriangleup \cdot \mathcal{R}_w(u,v) \cdot \bigtriangleup$ for elementary tensors and hence, by bilinearity, for all elements $u \in \mathcal{K}E, v \in \mathcal{K}F$. Since $\bigtriangleup$ is a unitary, Proposition 1 gives $(\|\mathcal{R}^{op})_w(v,u)\| = \|\mathcal{R}_w(u,v)\|$ for all $v \in \mathcal{K}F, u \in \mathcal{K}E$. The rest is clear. $\triangleright$

Again, column and raw Hilbertians provide several excellent illustrations.

Proposition 11. Every bounded bifunctional $f : H_r \times K_c \to \mathbb{C}$, where H and K are Hilbert spaces, is (automatically) strongly and hence weakly completely bounded. Moreover,

$$\|f\|_{scb} = \|f\|_{wcb} = \|f\|.$$

$\triangleleft$ By virtue of the definition of the column and row Hilbertians, elements of $\mathcal{K}K_c$ are identified with operators from $L = L \dot{\otimes} \mathbb{C}$ into $L \dot{\otimes} K$, and, in particular,

the elementary tensor ax transforms to the operator $\xi \mapsto a(\xi)\dot{\otimes}x$. At the same time elements of $\mathcal{K}H_r$ are identified with operators from $L\dot{\otimes}\overline{H}$ into $L = L\dot{\otimes}\mathbb{C}$, and the elementary tensor by transforms to the operator well defined by $\eta\dot{\otimes}z \mapsto b(\eta)\langle y, z\rangle$. (We emphasize that $\langle\cdot,\cdot\rangle$ denotes in our argument the inner product in H, and not in $\overline{H}$).

As is well known, our $f : H \times K \to \mathbb{C}$ gives rise to a bounded operator $\varphi : K \to \overline{H}$, well defined by $\langle y, \varphi(z)\rangle = f(y, z); y \in \overline{H}, z \in K$, and we have $\|f\| = \|\varphi\|$. Consider, for $u \in \mathcal{K}K$ and $v \in \mathcal{K}H$ in their capacity of operators, the diagram

$$L \xrightarrow{u} L\dot{\otimes}K \xrightarrow{\mathbf{1}\dot{\otimes}\varphi} L\dot{\otimes}\overline{H} \xrightarrow{v} L.$$

If $u = ax$ and $v = by$, then the easy calculation shows that the respective operator composition takes $\xi \in L$ to $f(y, x)ba(\xi)$, that is to $[f_s(v, u)](\xi)$. (Here $\mathcal{K}\otimes\mathbb{C}$, the range of f_s, is, of course, identified with $\mathcal{K}$). By bilinearity, we have that our composition is $f_s(v, u)$ for all $v \in \mathcal{K}H$ and $u \in \mathcal{K}K$. But then $\|f_s(v, u)\| \leq \|v\|\|\mathbf{1}\dot{\otimes}\varphi\|\|u\| = \|f\|\|v\|\|u\|$. The rest is clear. ▷

Combining this proposition with Proposition 10, we get

Corollary 1. Every bounded bifunctional $f : H_c \times K_r \to \mathbb{C}$, where H and K are Hilbert spaces, is (automatically) weakly completely bounded, and $\|f\|_{wcb} = \|f\|$.

But why only weakly? Now the time of counter-examples arrived. Probably, the most transparent of them are based on the bifunctional of inner product $\langle\cdot,\cdot\rangle : H \times \overline{H} \to \mathbb{C}$.

It is easy to see that the strong amplification of this bifunctional takes the pair (ω, ϖ), introduced in Section 2, to the operator nP which has, of course, the norm n. Since we can take an arbitrary n, Proposition 6 implies that our bifunctional, being considered on $H_c \times \overline{H}_r$ is not strongly completely bounded. But we already know that such a bifunctional is weakly completely bounded. This shows, first, that the words "strong" and "weak" used here are not for nothing, and, second, that a bioperator $\mathcal{R}$ can well be strongly completely bounded whereas $\mathcal{R}^{op}$ is not.

Finally, to display a bounded bioperator which is not even weakly completely bounded, one can take the same bifunctional but considered on $H_c \times \overline{H}_c$ or on $H_r \times \overline{H}_r$. Indeed, the weak amplification of our bifunctional obviously takes the pair (ω, ω) to $\sum_{k=1}^n q_k^* \diamondsuit q_k^*$ and (ϖ, ϖ) to $\sum_{k=1}^n q_k \diamondsuit q_k$. In both cases, as is easy to see, we get an operator of norm $\sqrt{n}$. Again, Proposition 6 immediately gives what we want.

One more example, this time of more general nature, deserves our special attention. Suppose that E and F are explicitly presented as operator spaces. Consider their spatial tensor product $E\dot{\otimes}F$ (see Section 1) and equip it with the standard quantum norm. This means, as we remember, that $\mathcal{K}(E\dot{\otimes}F)$ is identified with the operator space $\mathcal{K}\dot{\otimes}(E\dot{\otimes}F)$ (as well as $\mathcal{K}E = \mathcal{K}\dot{\otimes}E$ and $\mathcal{K}F = \mathcal{K}\dot{\otimes}F$).

Proposition 12. *The bioperator $\mathcal{T} : E \times F \to E\dot{\otimes}F$, acting as $(x, y) \mapsto x\dot{\otimes}y$, is strongly completely contractive.*

◁ Consider $\mathcal{T}_s : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}\dot{\otimes}(E\dot{\otimes}F)$, in the current situation acting between $(\mathcal{K}\dot{\otimes}E) \times (\mathcal{K}\dot{\otimes}F)$ and $\mathcal{K}\dot{\otimes}(E\dot{\otimes}F)$. Let E is presented as a subspace of $\subseteq \mathcal{B}(H_1, K_1)$, and F as that of $\subseteq \mathcal{B}(H_2, K_2)$. Then we have the inclusions

$E\dot{\otimes}F \subseteq \mathcal{B}(H_1\dot{\otimes}H_2, K_1\dot{\otimes}K_2), \mathcal{K}\dot{\otimes}E \subseteq \mathcal{B}(L\dot{\otimes}H_1, L\dot{\otimes}K_1), \mathcal{K}\dot{\otimes}F \subseteq \mathcal{B}(L\dot{\otimes}H_2, L\dot{\otimes}K_2)$ and $\mathcal{K}\dot{\otimes}(E\dot{\otimes}F) \subseteq \mathcal{B}(L\dot{\otimes}(H_1\dot{\otimes}H_2), L\dot{\otimes}(K_1\dot{\otimes}K_2))$.

Take $u \in \mathcal{K}\dot{\otimes}E$ and $v \in \mathcal{K}\dot{\otimes}F$. Introduce the operators $U := u\dot{\otimes}\mathbf{1}_{K_2} : L\dot{\otimes}H_1\dot{\otimes}K_2 \to L\dot{\otimes}K_1\dot{\otimes}K_2$ and $V : L\dot{\otimes}H_1\dot{\otimes}H_2 \to L\dot{\otimes}H_1\dot{\otimes}K_2$ that coincides with $v\dot{\otimes}\mathbf{1}_{H_1}$ after the natural identification of Hilbert tensor products with factors presented in different order. We see that the composition UV maps $L\dot{\otimes}H_1\dot{\otimes}H_2$ into $L\dot{\otimes}K_1\dot{\otimes}K_2$.

Now assume, for a moment, that u and v are elementary tensors, say ax and by. Then UV obviously takes the elementary tensor $\xi\otimes\eta\otimes\zeta \in L\dot{\otimes}H_1\dot{\otimes}H_2$ to $ab(\xi)\otimes x(\eta)\otimes y(\zeta) \in L\dot{\otimes}K_1\dot{\otimes}K_2$. This means that in the considered case UV, after the installing of the proper brackets in the respective Hilbert tensor products, is not other thing than $\mathcal{T}_s(u,v)$. It follows, by the bilinearity of the relevant operations, that the same is true for all u and v. Consequently we have

$$\|\mathcal{T}_s(u,v)\| = \|UV\| \leq \|U\|\|V\| = \|u\dot{\otimes}\mathbf{1}_{K_2}\|\|v\dot{\otimes}\mathbf{1}_{H_1}\| = \|u\|\|v\|.$$

The rest is clear. $\triangleright$

4. The Haagerup tensor product

The role of tensor products in quantum functional analysis is even more important than in classical functional analysis. Their raison d'être is essentially the same: they "linearize" bilinear operators. As a "classical" prototype, both of our quantum tensor products have the projective tensor product of Grothendieck of normed spaces (cf., e.g., [**10**, Ch.2§7]). In fact we shall show that their constructions are slight complifications of the construction of the projective tensor product (see our introduction).

Up to the rest of our paper, we fix arbitrary quantum spaces E and F.

Definition 9 (cf. "classical" Definition 2.8.3 *idem*). We say that the pair (Θ,θ), consisting of a quantum space Θ and a strongly (respectively, weakly) completely contractive operator $\theta : E\times F \to \Theta$, is the *Haagerup tensor product* (respectively, the *four-named tensor product*)[3], if, for every strongly (respectively, weakly) completely contractive bioperator $\mathcal{R} : E\times F \to G$, where G is some third quantum space, there exists a completely contractive operator $R : \Theta \to G$ such that the diagram

$$\begin{tikzcd} E\times F \arrow[d, "\theta"'] \arrow[dr, "\mathcal{R}"] & \\ \Theta \arrow[r, "R"] & G \end{tikzcd}$$

[3]The origin of the first term is explained, e.g., in [**7**, p.173]. Our "four names" are those of Effros/Ruan and Blecher/Paulsen who have discovered the notion (in the "matricial" presentation) simultaneously and independently in [**13**] and [**14**],1991. This second version of the tensor product is called just projective tensor product in many papers and textbooks, notably in [**7**]. We feel, however, that in our "non-coordinate" presentation these words could create some confusion with the classical meaning of the term, due to Grothendieck. Moreover, if we compare the respective quantum norms, the first (Haagerup) tensor product is, to speak informally, even "more projective" than the second one (see Theorem 3 below).

is commutative.

This (so far hypothetical) operator R is called *associated with the bioperator* $\mathcal{R}$.

Remark. We emphasize that normed spaces considered in this paper, are not, generally speaking, assumed to be complete. We only note that there exists a substantial notion of a quantum *Banach* space, and both mentioned types of quantum tensor products have their Banach versions. The latter can be constructed with the help of the proper quantum version of the classical construction of the completion of a normed space. But we do not touch this circle of questions in this paper.

Using the standard general-categorical argument, based on the uniqueness of the initial object in a category, one can easily prove that, for each of our versions of the quantum tensor product, the relevant uniqueness theorem is valid. Namely, if $(\Theta_k, \theta_k); k = 1, 2$ are two Haagerup (respectively, four-named) tensor products of E and F, then there exists a completely isometrical isomorphism $I : \Theta_1 \to \Theta_2$, making the diagram

$$\begin{tikzcd} & E \times F \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] & \\ \Theta_1 \arrow[rr, "I"] & & \Theta_2 \end{tikzcd}$$

commutative. But we shall not do it here.

We proceed to the existence theorem for both types of quantum tensor products. This will be proved by displaying their explicit constructions. In both cases (just as in the classical case) our Θ, as a linear space, is the algebraic tensor product $E\otimes F$, and θ, as a bioperator, is the canonical bioperator $\vartheta : E\times F \to E\otimes F : (x, y) \mapsto x\otimes y$. Accordingly, our task is to supply the amplification $\mathcal{K}(E \otimes F)$ by two appropriate norms. (We recall again that in this paper we consider the "non-completed" versions of our tensor products; cf. the previous remark).

We begin with the Haagerup tensor product. Let us consider the strong amplification $\vartheta_s : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}(E \otimes F)$ of ϑ and use the notation $u \odot v$ instead of $\vartheta_s(u, v)$. We see that the operation $\odot$, the so-called "Effros symbol", is bilinear. (It is the non-coordinate analogue of the known "Effros symbol" in the matricial exposition; cf., e.g., [**7**, §9.1]). Thus it is well-defined on elementary tensors by $ax \odot by = (ab)(x \otimes y)$. Note also that, for every $a \in \mathcal{B}, u \in \mathcal{K}E, v \in \mathcal{K}F$ we have $u \cdot a \odot v = u \odot a \cdot v$. (In other words, the bioperator $\odot$ is balanced with respect to the right outer multiplication in $\mathcal{K}E$ and the left outer multiplication in $\mathcal{K}F$).

Let $\odot : \mathcal{K}E \otimes \mathcal{K}F \to \mathcal{K}(E \otimes F)$ be the linear operator, associated with ϑ_s; it is well-defined by $\odot(u \otimes v) := u \odot v$. Since every element of $\mathcal{K}$ is a product of other elements, every elementary tensor and hence arbitrary element in $\mathcal{K}(E \otimes F)$ belongs to the image of $\odot$; in other words, $\odot$ is surjective (Soon, in Proposition 16, we shall see that the same is true even for the map ϑ_s itself). Therefore $\mathcal{K}(E \otimes F)$ can be identified with a quotient space of $\mathcal{K}E \otimes \mathcal{K}F$. Consequently, every norm on $\mathcal{K}E \otimes \mathcal{K}F$ gives rise to its quotient semi-norm on $\mathcal{K}(E \otimes F)$, and the latter semi-norm is uniquely determined by the claim that $\odot$ is a coisometric operator (i.e. it maps the open unit ball of $\mathcal{K}E \otimes \mathcal{K}F$ onto that of $\mathcal{K}(E \otimes F)$.

Now we apply this construction to the projective norm $\|\cdot\|_p$ in $\mathcal{K}E\otimes\mathcal{K}F$, that is to the norm of the (non-completed) Grothendieck projective tensor product of normed spaces $\mathcal{K}E$ and $\mathcal{K}F$. (We recall, that, for given normed spaces X and Y, and $u\in X\otimes Y$, $\|u\|_p:=\inf\{\sum_{k=1}^n\|x_k\|\|y_k\|\}$, where the infimum is taken for all possible representations of u as $\sum_{k=1}^n x_k\otimes y_k; x_k\in X, y_k\in Y$; cf., e.g., [**10**, Ch.2§7]. *The resulting quotient semi-norm on* $\mathcal{K}(E\otimes F)$ *is denoted by* $\|\cdot\|_h$.[4] Thus, for every $U\in\mathcal{K}(E\otimes F)$ we have

$$\|U\|_h:=\inf\{\sum_{k=1}^n\|u_k\|\|v_k\|\},\tag{6}$$

where the infimum is taken for all possible representations of U as $\sum_{k=1}^n u_k\odot v_k; u_k\in\mathcal{K}E, v_k\in\mathcal{K}F$.

Note that $\mathcal{K}E\otimes\mathcal{K}F$, being a (non-module) tensor product of a left and a right $\mathcal{B}$-bimodules, has the canonical structure of a $\mathcal{B}$-bimodule, and $\odot$ is obviously a morphism of respective $\mathcal{B}$-bimodules.

Proposition 13. *The semi-norm* $\|\cdot\|_h$ *in the* $\mathcal{B}$*-bimodule* $\mathcal{K}(E\otimes F)$ *satisfies the first axiom of Ruan.*

$\triangleleft$ It is well known and easy to check that, for a normed algebra A, the (non-module) projective tensor product of a left contractive normed A-module and a right contractive normed A-module is a contractive A-bimodule. This concerns, in particular, the $\mathcal{B}$-bimodule $(\mathcal{K}E\otimes\mathcal{K}F,\|\cdot\|_p)$. But it was observed that $\mathcal{K}(E\otimes F)$ is the image of the latter bimodule with respect to a coisometric $\mathcal{B}$-bimodule morphism. The rest is clear. $\triangleright$

Proposition 14. *Let G be a quantum space, $\mathcal{R}:E\times F\to G$ a strongly completely bounded bioperator, $R:E\otimes F\to G$ the associated linear operator. Then the amplification $R_\infty:\mathcal{K}(E\otimes F)\to\mathcal{K}G$ is a bounded operator with respect to the semi-norm $\|\cdot\|_h$ and the given quantum norm on G. Moreover, $\|R_\infty\|=\|\mathcal{R}\|_{scb}$.*

$\triangleleft$ Consider the diagram

$$\begin{tikzcd}
\mathcal{K}E\otimes\mathcal{K}F \arrow[d, "\odot"] \arrow[dr, "R^s"] & \\
\mathcal{K}(E\otimes F) \arrow[r, "R_\infty"] & \mathcal{K}G
\end{tikzcd}$$

where R^s is the operator, associated with the strong amplification $\mathcal{R}_s:\mathcal{K}E\times\mathcal{K}F\to\mathcal{K}G$ of the bioperator $\mathcal{R}$. By the universal property of the projective norm, we have $\|\mathcal{R}\|_{scb}=\|R^s\|$. Further, routine calculations with elementary tensors in $\mathcal{K}E\otimes\mathcal{K}F$ show that this diagram is commutative. It follows, taking into account that $\odot$ is a coisometric operator, that $\|R_\infty\|=\|R^s\|$. The rest is clear. $\triangleright$

Proposition 15. (As a matter of fact) $\|\cdot\|_h$ *is a norm.*

$\triangleleft$ Combining Propositions 4 and 13, we see that it is sufficient to show that, for a non-zero elementary tensor $aw; a\in\mathcal{K}, w\in E\otimes F$ we have $\|aw\|_h>0$. Since $w\neq 0$, it is well known (cf.the proof of [**10**, Proposition 2.7.6]) that there exist bounded functionals $f:E\to\mathbb{C}$ and $g:F\to\mathbb{C}$ such that $(f\otimes g)w\neq 0$. Now put in

[4]For all evidence, the subindex "h" is used in the literature to honour Uffe Haagerup. Well, "H" is everywhere reserved for Hilbert...

the previous proposition $\mathcal{R} := f \times g : E \times F \to \mathbb{C}$. By virtue of Proposition 9, $\mathcal{R}$ is strongly completely bounded, and $\|\mathcal{R}\|_{scb} = \|f\|\|g\|$. Since in our case $R = f \otimes g$, Proposition 14 gives $\|R_\infty\| = \|f\|\|g\|$. But, obviously, $R_\infty(aw) = [(f \otimes g)(w)]a$. From this, since $a \neq 0$ and $(f \otimes g)w \neq 0$, we have $\|R_\infty\|\|aw\|_h \geq \|R_\infty(aw)\| > 0$. The rest is clear. $\triangleright$

¿From now on, we shall call $\| \cdot \|_h$ *Haagerup norm.*

Now we shall show that the expression (6) for the Haagerup norm can be simplified. Apart from the independent interest, this will help to shorten some further proofs.

Proposition 16. *Every $U \in \mathcal{K}(E \otimes F)$ can be represented as* (a single "Effros symbol") *$u \odot v; u \in \mathcal{K}E, v \in \mathcal{K}F$. Moreover, we have*

$$\|U\|_h := \inf\{\|u\|\|v\|\}, \tag{7}$$

where the infimum is taken for all possible representations of U in the indicated form.

$\triangleleft$ Take $\varepsilon > 0$. Because of the equality (6), there exists a representation $U = \sum_{k=1}^n u_k \odot v_k$ such that $\sum_{k=1}^n \|u_k\|\|v_k\| < \|U\|_h + \varepsilon$. Choose isometric operators $S_1, ..., S_n \in \mathcal{B}$ with pairwise orthogonal images and put $u := \sum_{k=1}^n u_k \cdot S_k^*, v := \sum_{k=1}^n S_k \cdot v_k$. Then, since the bioperator ϑ_s is balanced and $S_k^* S_l = \delta_l^k \mathbf{1}$, we have $u \odot v = \sum_{k,l=1}^n u_k S_k^* S_l \odot v_l = \sum_{k=1}^n u_k \odot v_k = U$.

Further, pairwise orthogonal projections $S_k S_k^*$ are right supports of $u_k \cdot S_k^*$ and left supports of $S_k \cdot v_k; k = 1, ..., n$. Therefore, by virtue of Proposition 2, we have $\|u\| \leq (\sum_{k=1}^n \|u_k\|^2)^{\frac{1}{2}}$ and $\|v\| \leq (\sum_{k=1}^n \|v_k\|^2)^{\frac{1}{2}}$. But using, if necessary, scalar multiples, we have a right to assume that $\|u_k\| = \|v_k\|$ for all k. It follows that $\|u\|\|v\| \leq \sum_{k=1}^n \|u_k\|^2 < \|U\|_h + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the infimum in (7) is not bigger than $\|U\|_h$. The inverse inequality is obvious. $\triangleright$

Proposition 17. *The Haagerup norm on $\mathcal{K}(E \otimes F)$ satisfies the second axiom of Ruan.*

$\triangleleft$ Let $U, V \in \mathcal{K}(E \otimes F)$ have orthogonal supports P_1 and P_2. Obviously, we have a right to assume that $\|U\|_h > \|V\|_h$.

Using the previous proposition, take ε with $0 < \varepsilon < \|U\|_h - \|V\|_h$ and representations $U = u_1 \odot v_1$, $V = u_2 \odot v_2$ such that $\|u_1\|\|v_1\| < \|U\|_h + \varepsilon$ and $\|u_2\|\|v_2\| < \|V\|_h + \varepsilon$. We can, of course, assume that $u_1 = P_1 \cdot u_1, u_2 = P_2 \cdot u_2, v_1 = v_1 \cdot P_1, v_2 = v_2 \cdot P_2$ and also $\|u_1\| \geq \|u_2\|$ and $\|v_1\| \geq \|v_2\|$. Finally, using the same device as at the end of the proof of Proposition 2, we can assume that the images of P_1 and P_2 are infinite-dimensional.

Now take isometric operators S_k with $S_k S_k^* = P_k; k = 1, 2$ and put $u := u_1 \cdot S_1^* + u_2 \cdot S_2^*$, $v := S_1 \cdot v_1 + S_2 \cdot v_2$. Routine calculations show that $u \odot v = U + V$ and hence $\|U + V\|_h \leq \|u\|\|v\|$. But we have $u_k \cdot S_k^* = P_k \cdot (u_k \cdot S_k^*) \cdot P_k; k = 1, 2$. Therefore, because the norm on $\mathcal{K}E$ satisfies (RII), $\|u\| = \max\{\|u_1 \cdot S_1^*\|, \|u_2 \cdot S_2^*\|\}$ and hence, by Proposition 1, $\|u\| = \max\{\|u_1\|, \|u_2\|\} = \|u_1\|$. Similarly we have $\|v\| = \|v_1\|$. Consequently $\|U + V\|_h \leq \|u_1\|\|v_1\| < \|U\|_h + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\|U + V\|_h \leq \|U\|_h$. Because of $P_1 \cdot (U + V) = U$, the inverse inequality follows from (RI). The rest is clear. $\triangleright$

Combining Propositions 13 and 17, we obtain

Corollary 2. *The Haagerup norm is a quantum norm on $E \otimes F$.*

We denote the constructed quantum space by $E \otimes_h F$. The same symbol will denote the underlying normed space; this will not lead to a misunderstanding.

Theorem 2. *Let G be an arbitrary quantum space, and $\mathcal{R} : E \times F \to G$ an arbitrary strongly completely bounded bioperator. Then there exists a unique completely bounded operator $R : E \otimes_h F \to G$ such that the diagram*

$$\begin{tikzcd} E \times F \arrow[d, "\vartheta"] \arrow[dr, "\mathcal{R}"] & \\ E \otimes_h F \arrow[r, "R"] & G \end{tikzcd}$$

is commutative. Moreover, we have $\|R\|_{cb} = \|\mathcal{R}\|_{scb}$.

$\triangleleft$ Pure algebra provides a unique linear operator R, making our diagram commutative. The rest follows from Proposition 14. $\triangleright$

Note that, for $u \in \mathcal{K}E, v \in \mathcal{K}F$, $\|\vartheta_s(u,v)\|_h = \|u \bigodot v\|_h \le \|u\|\|v\|$, and this means that $\vartheta : E \times F \to E \otimes_h F$ is strongly completely contractive. Therefore the previous theorem implies

Corollary 3 ("the existence theorem"). *The pair $(E \otimes_h F, \vartheta)$ is the Haagerup tensor product of quantum spaces E and F.*

We proceed to the realization of Haagerup tensor product as a projective tensor product of normed modules.

Take, for a moment, an arbitrary normed algebra A, a right normed A-module X, and a left normed A-module Y. Recall that, by definition, the projective tensor product of these modules is their algebraic module tensor product $X \underset{A}{\otimes} Y$, equipped by the special semi-norm $\|\cdot\|_{mp}$. The latter is the quotient semi-norm of the projective norm $\|\cdot\|_p$ in $X \otimes Y$ with respect to the canonical quotient map $\tau : X \otimes Y \to X \underset{A}{\otimes} Y$. (The operator τ is well defined by taking $x \otimes y$ to $x \underset{A}{\otimes} y$). In other words, for $U \in X \underset{A}{\otimes} Y$, $\|U\|_{mp} = \inf\{\sum_{k=1}^n \|u_k\|\|v_k\|\}$, where the infimum is taken for all possible representations $U = \sum_{k=1}^n u_k \underset{A}{\otimes} v_k; u_k \in X, v_k \in Y$. If X and Y are not only one-sided modules but A-bimodules,then $X \underset{A}{\otimes} Y$ also becomes an A-bimodule with outer multiplications well defined by $a \cdot (u \otimes_A v) := (a \cdot u) \otimes_A v$ and $(u \underset{A}{\otimes} v) \cdot b := u \underset{A}{\otimes} (v \cdot b)$. Moreover, if we consider $X \otimes Y$ as a tensor product of left and right A-modules, τ becomes a morphism of A-bimodules.

In our special context $A = \mathcal{B}$, $X = \mathcal{K}E$ and $Y = \mathcal{K}F$. As it was mentioned, the strong amplification $\vartheta_s : \mathcal{K}E \times \mathcal{K}F \to \mathcal{K}(E \otimes F)$ is a balanced bioperator. Therefore, by the universal property of the module tensor product, it gives rise to the linear operator $\odot_{\mathcal{B}} : \mathcal{K}E \underset{\mathcal{B}}{\otimes} \mathcal{K}F \to \mathcal{K}(E \otimes F)$, well defined by $u \underset{\mathcal{B}}{\otimes} v \mapsto \vartheta_s(u,v)$.

Theorem 3. *The operator $\odot_{\mathcal{B}}$ is an isometric $\mathcal{B}$-bimodule isomorphism between* $(\mathcal{K}E \underset{\mathcal{B}}{\otimes} \mathcal{K}F, \|\cdot\|_{mp})$ and $\mathcal{K}(E \otimes_h F)$.

$\triangleleft$ Consider the diagram

$$\begin{tikzcd}
(\mathcal{K}E \otimes \mathcal{K}F, \|\cdot\|_p) \arrow[d, "\tau"] \arrow[dr, "\odot"] & \\
(\mathcal{K}E \underset{\mathcal{B}}{\otimes} \mathcal{K}F, \|\cdot\|_{mp}) \arrow[r, "\odot_{\mathcal{B}}"] & \mathcal{K}(E \otimes_h F)
\end{tikzcd}$$

Recall that $\odot$ and τ are morphisms of $\mathcal{B}$-bimodules, and both of them, by definition of $\|\cdot\|_h$ and $\|\cdot\|_{mp}$, are coisometric operators. Since the diagram is obviously commutative, the operator $\odot_{\mathcal{B}}$ also has both properties. Therefore all what we need is to show that the latter operator is injective.

At first consider the simplest particular case where $E = F = \mathbb{C}$. Then $\odot_{\mathcal{B}}$ is, of course, just the so-called product map $\pi : \mathcal{K} \underset{\mathcal{B}}{\otimes} \mathcal{K} \to \mathcal{K} : a \underset{\mathcal{B}}{\otimes} b \mapsto ab$. Take $U \in \mathcal{K} \underset{\mathcal{B}}{\otimes} \mathcal{K}$; let it have the form $U = \sum_{k=1}^{n} a_k \underset{\mathcal{B}}{\otimes} b_k$ for some compact operators a_k, b_k. As is well known (and easy to check), there exist $c, d_k \in \mathcal{K}$ such that $a_k = cd_k$ for all k and c is a not a divisor of zero. Consequently $U = \sum_{k=1}^{n} c \underset{\mathcal{B}}{\otimes} d_k b_k = c \underset{\mathcal{B}}{\otimes} (\sum_{k=1}^{n} d_k b_k)$ and $\pi(U) = c(\sum_{k=1}^{n} d_k b_k)$. Therefore, if $\pi(U) = 0$, then $\sum_{k=1}^{n} d_k b_k = 0$ and hence $U = c \underset{\mathcal{B}}{\otimes} 0 = 0$. We see that π is injective (and thus it is an isometric isomorphism).

Now return to general E and F. "Changing the order of the relevant tensor factors", we easily see that the space $\mathcal{K}E \underset{\mathcal{B}}{\otimes} \mathcal{K}F$ coincides, up to a linear isomorphism, with $(\mathcal{K} \underset{\mathcal{B}}{\otimes} \mathcal{K}) \otimes (E \otimes F)$. (To be precise, this isomorphism and its inverse are defined, by an obvious way, with the help of the 4-linear operators, acting as $(a, x, b, y) \mapsto (a \underset{\mathcal{B}}{\otimes} b) \otimes (x \otimes y)$ and $(a, b, x, y) \mapsto ax \underset{\mathcal{B}}{\otimes} by; a, b \in \mathcal{K}, x \in E, y \in F$). Moreover, under such an identification, the operator $\odot_{\mathcal{B}}$ transforms to $\pi \otimes \mathbf{1} : (\mathcal{K} \underset{\mathcal{B}}{\otimes} \mathcal{K}) \otimes (E \otimes F) \to \mathcal{K}(E \otimes F)$. But we know that π is injective. Hence the same is true for $\pi \otimes \mathbf{1}$. The rest is clear. $\triangleright$

If E and F are operator spaces, we can identify $E \otimes F$ with $E \dot{\otimes} F$ and compare Haagerup quantum norm with the standard quantum norm. The latter will be denoted by $\|\cdot\|_{sp}$ and the respective quantum space by $E \otimes_{sp} F$.

Proposition 18. *Let E and F be operator spaces. Then we have $\|\cdot\|_{sp} \le \|\cdot\|_h$.*

$\triangleleft$ By Proposition 12, the bioperator $\vartheta : E \times F \to E \otimes_{sp} F$ is strongly completely contractive. Therefore the definition of the Haagerup tensor product gives, with ϑ as $\mathcal{R}$, that $\mathbf{1} : E \otimes_h F \to E \otimes_{sp} F$ is contractive. The rest is clear. $\triangleright$

5. The four-named tensor product

We turn to the explicit construction of the second principal quantum tensor product. Beginning with the same canonical bioperator ϑ, now we consider its weak amplification ϑ_w. Let us write $u \diamondsuit v$ instead of $\vartheta_w(u, v); u \in \mathcal{K}E, v \in \mathcal{K}F$. Of course, this extended "diamond operation" is well-defined by $ax \diamondsuit by = (a \diamondsuit b)(x \otimes y)$ and hence, by bilinearity, satisfy the identity

$$(a \diamondsuit b) \cdot (u \diamondsuit v) \cdot (c \diamondsuit d) = (a \cdot u \cdot c) \diamondsuit (b \cdot v \cdot d). \tag{8}$$

Let $\diamondsuit : \mathcal{K}E \otimes \mathcal{K}F \to \mathcal{K}(E \otimes F)$ be the linear operator, associated with ϑ_w; it is well-defined by $\diamondsuit(u \otimes v) = u \diamondsuit v$. This operator, contrary to $\odot$, is not bound to be

surjective. However, another, slightly more complicated operator has this attractive property. We come to this operator after the following observation.

Proposition 19. *Every $a \in \mathcal{K}$ has the form $b(c\diamondsuit d)b'$ for some $b, b', c, d \in \mathcal{K}$, and even $b(c\diamondsuit c)b'$ for some $b, b', c \in \mathcal{K}$.*

$\triangleleft$ Fix an arbitrary orthonormal basis $e_n; n = 1, 2, ...$ in L and put $e_{m,n} := i^*(e_m \otimes e_n)$. It immediately follows from the classical Schmidt Theorem (see, e.g., [**10**, Theorem 2.4.1]) that a has a factorization IhJ where I, J are unitary operators on L, and h is a compact positive operator with eigenvectors $e_{m,n}$. Let $\lambda_{m,n}$ be the respective eigenvalues.

There exists, of course, a decreasing sequence $t_n \geq \max\{\lambda_{m,n}, \lambda_{n,m} : m = 1, ..., n\}$, converging to 0. Then the double sequence $r_{m,n} := \sqrt{t_m t_n}$ is not less than $\lambda_{m,n}$ and also converges to 0. Therefore $\lambda_{m,n} = r_{m,n} s_{m,n}^2$ for some non-negative $s_{m,n} \leq 1$. Consider the compact operator c' well defined by $e_n \mapsto \sqrt{t_n} e_n$ and the bounded operator f well defined by $e_{m,n} \mapsto s_{m,n} e_{m,n}$. It is easy to check that $(c'\diamondsuit c')e_{m,n} = r_{m,n} e_{m,n}$ and hence $f(c'\diamondsuit c')f(e_{m,n}) = h(e_{m,n})$. Hence $a = If(c'\diamondsuit c')fJ$. But c' factorizes as gcg' for some $c, g, g' \in \mathcal{K}$. Consequently $c'\diamondsuit c' = (g\diamondsuit g)(c\diamondsuit c)(g'\diamondsuit g')$, and it remains to put $b := If(g\diamondsuit g)$ and $b' := (g'\diamondsuit g')fJ$. $\triangleright$

Now we introduce the operator $\uplus : \mathcal{K} \otimes \mathcal{K}E \otimes \mathcal{K}F \otimes \mathcal{K} \to \mathcal{K}(E \otimes F)$, associated with the 4-linear operator $(b, u, v, d) \mapsto b \cdot (u\diamondsuit v) \cdot d$.

Proposition 20. *The operator $\uplus$ is surjective.*

$\triangleleft$ It follows from the previous proposition that an element in $\mathcal{K}(E \otimes F)$ of the form $a(x \otimes y)$ is equal to $b \cdot (c\diamondsuit c)(x \otimes y) \cdot d$, that is belongs to the image of $\uplus$. It remains to recall that an arbitrary element in $\mathcal{K}(E \otimes F)$ is a sum of several elements of the indicated form. $\triangleright$

Thus $\mathcal{K}(E \otimes F)$ can be identified with a quotient space of $\mathcal{K} \otimes \mathcal{K}E \otimes \mathcal{K}F \otimes \mathcal{K}$. Introduce on the latter space the projective norm $\| \cdot \|_p$ (that is, the projective tensor product of the four relevant norms), and consider the respective quotient semi-norm on $\mathcal{K}(E \otimes F)$. The latter will be denoted by $\| \cdot \|_4$. In other words, our semi-norm is defined by

$$\|U\|_4 := \inf\{\sum_{k=1}^{n} \|a_k\|\|u_k\|\|v_k\|\|b_k\|\}, \tag{9}$$

where the infimum is taken for all possible representations of U as $\sum_{k=1}^{n} a_k \cdot (u_k\diamondsuit v_k) \cdot b_k; a_k, b_k \in \mathcal{K}, u_k \in \mathcal{K}E, v_k \in \mathcal{K}F$. Note that, with respect to $\| \cdot \|_p$ and $\| \cdot \|_4$, the operator $\uplus$ is coisometric.

Proposition 21. *The semi-norm $\| \cdot \|_4$ in the $\mathcal{B}$-bimodule $\mathcal{K}(E \otimes F)$ satisfies the first axiom of Ruan.*

$\triangleleft$ The proof repeats, with obvious modifications, that of Proposition 13, and we omit it here. $\triangleright$

Proposition 22. *Let G be a quantum space, $\mathcal{R} : E \times F \to G$ a weakly completely bounded bioperator, $R : E \otimes F \to G$ the associated linear operator. Then the amplification $R_\infty : \mathcal{K}(E \otimes F) \to \mathcal{K}G$ is a bounded operator with respect to the semi-norm $\| \cdot \|_4$ and the quantum norm on G. Moreover, $\|R_\infty\| = \|\mathcal{R}\|_{wcb}$.*

$\triangleleft$ Consider the 4-linear operator $\mathcal{S} : \mathcal{K} \times \mathcal{K}E \times \mathcal{K}F \times \mathcal{K} \to \mathcal{K}G : (a, u, v, b) \mapsto a \cdot \mathcal{R}_w(u, v) \cdot b$. Since G satisfies (RI), we easily see that the weak complete boundedness of $\mathcal{R}$ implies the (usual) boundedness of $\mathcal{S}$, and $\|\mathcal{S}\| \leq \|\mathcal{R}_w\|$. At the same time we obviously have $\mathcal{R}_w(u, v) = \lim_{n\to\infty} \mathcal{S}(P_N, u, v, P_N) \in \mathcal{K}G$, where P_N is an approximate identity in $\mathcal{K}$ consisting of projections. Therefore the boundedness of $\mathcal{S}$ implies the complete boundedness of $\mathcal{R}$, and $\|\mathcal{R}_w\| \leq \|\mathcal{S}\|$. Thus both kinds of the boundedness are equivalent, and $\|\mathcal{S}\| = \|\mathcal{R}\|_{wcb}$.

Now consider the diagram

$$\begin{tikzcd} \mathcal{K}\otimes\mathcal{K}E\otimes\mathcal{K}F\otimes\mathcal{K} \arrow[d, "\uplus"] \arrow[dr, "S"] & \\ \mathcal{K}(E\otimes F) \arrow[r, "R_\infty"] & \mathcal{K}G \end{tikzcd}$$

where S is the operator, associated with the 4-linear operator $\mathcal{S}$. By the known property of the projective norm, we have $\|S\| = \|\mathcal{S}\|$. Further, routine calculations with elementary tensors in $\mathcal{K}\otimes\mathcal{K}E\otimes\mathcal{K}F\otimes\mathcal{K}$ show that this diagram is commutative. It follows, taking into account that $\uplus$ is a coisometric operator, that $\|R_\infty\| = \|S\|$. The rest is clear. $\triangleright$

Proposition 23. *The estimate $\|\cdot\|_h \leq \|\cdot\|_4$ is valid; as a corollary, $\|\cdot\|_4$ is a norm.*

$\triangleleft$ We know that the canonical bioperator $\vartheta : E \times F \to E \otimes F$ is strongly completely contractive with respect to quantum norms on E, F and the (quantum) Haagerup norm on $E\otimes F$. Hence, by Theorem 1, it is weakly completely contractive with respect to the same quantum norms. Put it as $\mathcal{R}$ in the previous proposition. In this situation R is, of course, the identity operator in $E \otimes F$, and R_∞ is the identity operator from $(\mathcal{K}(E \otimes F), \|\cdot\|_4)$ onto $(\mathcal{K}(E \otimes F), \|\cdot\|_h)$. By the same proposition, $\|R_\infty\| \leq 1$. The rest is clear. $\triangleright$

¿From now on, we shall call $\|\cdot\|_4$ the *four-named norm*.[5]

The role of the following observation concerning the introduced norm is similar to that of Proposition 16 for the Haagerup norm.

Proposition 24. *Every $U \in \mathcal{K}(E \otimes F)$ can be represented as* (a ”single rigged diamond”)

$$a \cdot (u \diamondsuit v) \cdot b$$

where $a, b \in \mathcal{K}, u \in \mathcal{K}E, v \in \mathcal{K}F$. In more detail, if $U = \sum_{k=1}^{n} a_k \cdot (u_k \diamondsuit v_k) \cdot b_k; a_k, b_k \in \mathcal{K}, u_k \in \mathcal{K}E, v_k \in \mathcal{K}F$, and $S_1, ..., S_n$ are some isometric operators with pairwise orthogonal images, then, to obtain such a representation, one can take $a := \sum_{k=1}^{n} a_k(S_k^* \diamondsuit S_k^*), u := \sum_{k=1}^{n} S_k \cdot u_k \cdot S_k^*, v := \sum_{k=1}^{n} S_k \cdot v_k \cdot S_k^*$, and $b := \sum_{k=1}^{n}(S_k \diamondsuit S_k)b_k$. Finally, we have

$$\|U\|_4 := \inf\{\|a\|\|u\|\|v\|\|b\|\}, \tag{10}$$

where the infimum is taken for all possible representations of U in the indicated form.

$\triangleleft$ Recall that $S_k^* S_l = \delta_l^k$. Therefore the routine calculation using the equalities (3) and (8) shows that U indeed has the desired representation.

[5]”Blecher/Paulsen-Effros/Ruan norm”; cf. above.

To obtain the desired equality for $\|U\|_4$, take $\varepsilon > 0$. By (9), there exists a representation $U = \sum_{k=1}^n a_k \cdot (u_k \diamondsuit v_k) \cdot b_k$ such that $\sum_{k=1}^n \|a_k\| \|u_k\| \|v_k\| \|b_k\| < \|U\|_4 + \varepsilon$. Using, if necessary, scalar multiples, we have a right to assume that $\|a_k\| = \|b_k\|$ and $\|u_k\| = \|v_k\| = 1$ for all k, and thus $\sum_{k=1}^n \|a_k\|^2 < \|U\|_4 + \varepsilon$. Now take the representation of U as a "single rigged diamond", indicated above. We see that u is the sum of several elements of norm 1 with the pairwise orthogonal supports, namely $S_k S_k^*$, and the same is true for v. Therefore, by (RII), we have $\|u\| = \|v\| = 1$. Finally, the operator C^*-identity, together with the formula (3), gives

$$\|a\| = \left\| \left[\sum_{k=1}^n a_k (S_k^* \diamondsuit S_k^*) \right] \left[\sum_{l=1}^n (S_l \diamondsuit S_l) a_l^* \right] \right\|^{\frac{1}{2}} =$$

$$\left\| \sum_{k,l=1}^n a_k [(S_k^* S_l) \diamondsuit (S_k^* S_l)] a_l^*] \right\|^{\frac{1}{2}} = \left\| \sum_{k=1}^n a_k a_k^* \right\|^{\frac{1}{2}} \leq \left(\sum_{k=1}^n \|a_k\|^2 \right)^{\frac{1}{2}}.$$

Similar calculations, combined with $\|a_k\| = \|b_k\|$, give the same estimation for $\|b\|$. Consequently, $\|a\| \|u\| \|v\| \|b\| \leq \sum_{k=1}^n \|a_k\|^2 < \|U\|_4 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that the infimum in (10) is not bigger than $\|U\|_4$. The inverse inequality is obvious. $\triangleright$

Proposition 25. *The four-named norm on $\mathcal{K}(E \otimes F)$ satisfies the second axiom of Ruan.*

$\triangleleft$ Let $U, V \in \mathcal{K}(E \otimes F)$ have orthogonal supports P_1 and P_2. Again (as in the proof of Proposition 17) we have a right to assume that $\|U\|_4 > \|V\|_4$ and that the images of P_1 and P_2 are infinite-dimensional.

Using the previous proposition, take an arbitrary ε with $0 < \varepsilon < \|U\|_4 - \|V\|_4$ and representations $U = a_1 \cdot (u_1 \diamondsuit v_1) \cdot b_1$ and $V = a_2 \cdot (u_2 \diamondsuit v_2) \cdot b_2$ such that $\|a_1\| \|u_1\| \|v_1\| \|b_1\| < \|U\|_4 + \varepsilon$ and $\|a_2\| \|u_2\| \|v_2\| \|b_2\| < \|V\|_4 + \varepsilon$. Of course, we can assume that $a_k = P_k a_k, b_k = b_k P_k, \|u_k\| = \|v_k\| = 1; k = 1, 2$, and also $\|a_1\| \geq \|a_2\|$ and $\|b_1\| \geq \|b_2\|$. In particular, we have $\|a_1\| \|b_1\| < \|U\|_4 + \varepsilon$.

Now take isometric operators $S_k; k = 1, 2$ with $S_k S_k^* = P_k$ and put $a = a_1(S_1^* \diamondsuit S_1^*) + a_2(S_2^* \diamondsuit S_2^*), u = S_1 \cdot u_1 \cdot S_1^* + S_2 \cdot u_2 \cdot S_2^*, v = S_1 \cdot v_1 \cdot S_1^* + S_2 \cdot v_2 \cdot S_2^*$, and $b = (S_1 \diamondsuit S_1) b_1 + (S_2 \diamondsuit S_2) b_2$. By the part of Proposition 24, presenting the récipe of a "single diamond", we have

$$U + V = a \cdot (u \diamondsuit v) \cdot b.$$

It follows that $\|U + V\|_4 \leq \|a\| \|u\| \|v\| \|b\|$. Since the elements $S_k \cdot u_k \cdot S_k^*; k = 1, 2$ have supports P_k, the axioms (RII) and then (RI) give $\|u\| = \max\{\|S_k \cdot u_k \cdot S_k^*\|; k = 1, 2\} = 1$. Similarly $\|v\| = 1$. Finally, the operators $a_k(S_k^* \diamondsuit S_k^*); k = 1, 2$ have orthogonal left supports $P_k; k = 1, 2$ and, by (3), orthogonal right supports $P_k \diamondsuit P_k; k = 1, 2$.

As is well known in operator theory, these properties of the summands of a imply that $\|a\| = \max\{\|a_k(S_k^* \diamondsuit S_k)\|; k = 1, 2\}$. Hence, by Proposition 1, we have $\|a\| = \max\{\|a_k\|; k = 1, 2\} = \|a_1\|$. Similarly $\|b\| = \|b_1\|$. Therefore $\|U + V\|_4 \leq \|a_1\| \|b_1\| < \|U\|_4 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\|U + V\|_4 \leq \|U\|_4$. Because of $P_1 \cdot (U + V) = U$, the inverse inequality follows from (RI). The rest is clear. $\triangleright$

Combining Propositions 21 and 25, we obtain

Corollary 4. *The four-named norm is a quantum norm on $E \otimes F$.*

We denote the constructed quantum space by $E \otimes_4 F$. The same symbol ("the sign of the four") will be denote the underlying normed space; this will not lead to a misunderstanding.

Theorem 4. *Let G be an arbitrary quantum space, and $\mathcal{R} : E \times F \to G$ an arbitrary weakly completely bounded bioperator. Then there exists a unique completely bounded operator $R : E \otimes_4 F \to G$ such that the diagram*

$$\begin{array}{ccc} E \times F & & \\ \downarrow \vartheta & \searrow^{\mathcal{R}} & \\ E \otimes_4 F & \xrightarrow{R} & G \end{array}$$

is commutative. Moreover, we have $\|R\|_{cb} = \|\mathcal{R}\|_{wcb}$.

$\triangleleft$ The argument of Theorem 2 works, with Proposition 22 replacing Proposition 14. $\triangleright$

Note that, for $u \in \mathcal{K}E, v \in \mathcal{K}F$, we have $\|\vartheta_w(u, v)\|_4 = \|u \Diamond v\|_4 \le \|u\|\|v\|$. This means that $\vartheta : E \times F \to E \otimes_4 F$ is weakly completely contractive. Therefore the previous theorem implies

Corollary 5 ("the existence theorem"). *The pair $(E \otimes_4 F, \vartheta)$ is the four-named tensor product of quantum spaces E and F.*

6. Examples

Again, to get instructive illustrations, we turn to our well-beloved column and raw Hilbertians. Recall the identifications of H_c with $\mathcal{B}(\mathbb{C}, H)$ and of H_r with $\mathcal{B}(\overline{H}, \mathbb{C})$. In what follows, the symbols H_c and H_r denote, depending on the context, the respective standard quantum spaces or their underlying normed spaces; this will not lead to a confusion.

Proposition 26. *Let H be a Hilbert space, E an arbitrary operator space. Then, up to complete isometric isomorphisms, $H_c \otimes_h E = H_c \otimes_{sp} E$ and $E \otimes_h H_r = E \otimes_{sp} H_r$. More precisely, the identity operators $\mathbf{1} : H_c \otimes_h E \to H_c \otimes_{sp} E$ and $\mathbf{1} : E \otimes_h H_r \to E \otimes_{sp} H_r$ are complete isometric isomorphisms.*

$\triangleleft$ We already know, by Proposition 18, that both identity operators are completely contractive. Therefore our task is to show that their amplifications do not decrease norms.

Consider the "column" case. Take $U \in \mathcal{K}(H \otimes E)$. Identifying the latter space with $H \otimes (\mathcal{K}E)$ and using the first part of Proposition 5, we can represent U as $\sum_{k=1}^n e_k \otimes u_k$ with e_k as in that proposition and $u_k \in \mathcal{K}E$. (Of course, it does not matter that we have now the order of tensor factors different from that in the cited proposition).

Fix, for a time, an element of the form $\omega \in \mathcal{K}H_c$ indicated in (2) together with the relevant projection P and partial isometries $q_k; k = 1, ..., n$. Put $u := \sum_{k=1}^n q_k \cdot u_k$. Using that $q_k^* q_l = \delta_l^k P$, we have

$$\omega \odot u = \sum_{k,l=1}^n q_k^* e_k \odot q_l \cdot u_l = \sum_{k,l=1}^n q_k^* q_l e_k \odot u_l = \sum_{k=1}^n P e_k \odot u_k.$$

Assuming that u_k is an elementary tensor in $\mathcal{K}E$, one can easily check that $Pe_k \odot u_k \in \mathcal{K}(H \otimes E)$ is exactly $P \cdot (e_k \otimes u_k)$. Then, by bilinearity, the same is true in the general case. It obviously follows that $\omega \odot u = P \cdot U$.

By Proposition 6, we have $\|\omega\| = 1$. Further, elements $q_k \cdot u_k$ live in the operator space $\mathcal{K} \dot{\otimes} E$. Therefore, by C^*-identity we have

$$\|u\|_{sp} = \left\| \left(\sum_{k=1}^{n} q_k \cdot u_k \right)^* \left(\sum_{l=1}^{n} q_l \cdot u_l \right) \right\|^{\frac{1}{2}} = \left\| \sum_{k,l=1}^{n} [(q_k \dot{\otimes} \mathbf{1}) u_k]^* [(q_l \dot{\otimes} \mathbf{1}) u_l] \right\|^{\frac{1}{2}} =$$

$$\left\| \sum_{k,l=1}^{n} u_k^* [q_k^* q_l \dot{\otimes} \mathbf{1}] u_l \right\|^{\frac{1}{2}} = \left\| \sum_{k=1}^{n} u_k^* (P \dot{\otimes} \mathbf{1}) u_k \right\|^{\frac{1}{2}}.$$

Consequently, by the definition of the Haagerup norm, we have

$$\|P \cdot U\|_h \leq \|\omega\| \|u\| \leq \left\| \sum_{k=1}^{n} u_k^* (P \dot{\otimes} \mathbf{1}) u_k \right\|^{\frac{1}{2}}.$$

Now consider a sequence P_N of finite-dimensional projections, serving as an approximate identity in $\mathcal{K}$. Taking elementary tensors in $\mathcal{K} \dot{\otimes} E$ and using the bilinearity, we see that $\sum_{k=1}^{n} u_k^* (P_N \dot{\otimes} \mathbf{1}) u_k$ converges, with respect to the operator norm, to $\sum_{k=1}^{n} u_k^* u_k$. At the same time, of course, $P_N \cdot U$ converges to U in $\mathcal{K}(H_c \otimes_h E)$. Combined with the obtained inequality, both things give the estimate

$$\|U\|_h \leq \|\omega\| \|u\| \leq \| \sum_{k=1}^{n} u_k^* u_k \|^{\frac{1}{2}}.$$

Now turn to the norm of U as of an element of $\mathcal{K} \dot{\otimes} (H_c \dot{\otimes} E)$ or, equivalently, of $H_c \dot{\otimes} (\mathcal{K} \dot{\otimes} E)$. This time, by virtue of Proposition 5, we have the exact equality $\|U\|_{sp} = \| \sum_{k=1}^{n} u_k^* u_k \|^{\frac{1}{2}}$.

This ends the proof in the "column" case. The similar argument, with the obvious modifications, works in the "raw" case. $\triangleright$

Here is an illuminating particular case. Denote by $\mathcal{F}(H)$ the space of bounded finite-dimensional operators on H, endowed by the operator norm and the standard quantization.

Proposition 27. Let H be a Hilbert space. Then, up to a complete isometric isomorphism, $H_c \otimes_h \overline{H}_r = \mathcal{F}(H)$.

$\triangleleft$ By virtue of the previous proposition, it is sufficient to establish a complete isometric isomorphism between standard quantum spaces $H_c \otimes_{sp} \overline{H}_r$ and $\mathcal{F}(H)$.

The space $H_c \otimes_{sp} \overline{H}_r$ or, otherwise, $\mathcal{B}(\mathbb{C}, H) \dot{\otimes} \mathcal{B}(H, \mathbb{C})$, is a subspace in $\mathcal{B}(\mathbb{C} \dot{\otimes} H, H \dot{\otimes} \mathbb{C})$. The latter, because of the identification of $\mathbb{C} \dot{\otimes} H$ and $H \dot{\otimes} \mathbb{C}$ with H, coincides with $\mathcal{B}(H)$. It is easy to see that the resulting isometric embedding of $H_c \otimes_{sp} \overline{H}_r$ into $\mathcal{B}(H)$ takes an elementary tensor $x \dot{\otimes} y$ to the rank 1 operator $x \bigcirc y$. Obviously, the image of this embedding is $\mathcal{F}(H)$. Denote by $I : H_c \otimes_{sp} \overline{H}_r \to \mathcal{F}(H)$ the respective corestriction. Now it is sufficient for us to show that its amplification I_∞ is an isometric isomorphism.

Since our quantum spaces are standard, I_∞ acts between the subspace $\mathcal{K}\dot{\otimes} H_c\dot{\otimes}\overline{H}_r$ in $\mathcal{B}(L\dot{\otimes}\mathbb{C}\dot{\otimes}H, L\dot{\otimes}H\dot{\otimes}\mathbb{C})$ and the subspace $\mathcal{K}\dot{\otimes}\mathcal{F}(H)$ in $\mathcal{B}(L\dot{\otimes}H)$, and it is uniquely determined by taking $a\dot{\otimes}(x\dot{\otimes}y); a\in\mathcal{K}$ to $a\dot{\otimes}(x\bigcirc y)$. From this we easily see that I_∞ is a birestriction of a certain isometric isomorphism between these bigger spaces. The latter is generated by the natural identification of $L\dot{\otimes}\mathbb{C}\dot{\otimes}H$ and $L\dot{\otimes}H\dot{\otimes}\mathbb{C}$ with $L\dot{\otimes}H$. Therefore I_∞ is itself an isometric isomorphism. The rest is clear. $\triangleright$

We have described what happens if the left factor in the Haagerup tensor product is a column Hilbertian. But what if we put this Hilbertian on the right?

Proposition 28. *Let H be a Hilbert space, E an arbitrary quantum space. Then, up to complete isometric isomorphisms, $E\otimes_h H_c = E\otimes_4 H_c$ and $H_r\otimes_h E = H_r\otimes_4 E$. More precisely, the identity operators $\mathbf{1}: E\otimes_4 H_c\to E\otimes_h H_c$ and $\mathbf{1}: H_r\otimes_4 E\to H_r\otimes_h E$ are complete isometric isomorphisms.*

$\triangleleft$ By virtue of Proposition 23, our task is only to show that the amplifications of our identity operators do not decrease norms.

Consider the "column" case. Take $U\in\mathcal{K}(E\otimes H)$ and, using Proposition 16, represent it as a single Effros symbol $u\odot v; u\in\mathcal{K}E, v\in\mathcal{K}H$. Further, using Proposition 5, represent v as $\sum_{k=1}^n a_k e_k; a_k\in\mathcal{K}$, where e_k is an orthonormal system in L.

Taking some P and q_k, consider an element of the form ω as indicated in (2). Put $b := \sum_{k=1}^n a_k\diamondsuit q_k\in\mathcal{K}$. Then, using the equality (8), we have

$$(u\diamondsuit\omega)\cdot b=\sum_{k,l=1}^n (u\diamondsuit q_k^* e_k)\cdot(a_l\diamondsuit q_l)=\sum_{k,l=1}^n (u\cdot a_l)\diamondsuit(q_k^* q_l)e_k=\sum_{k=1}^n (u\cdot a_k)\diamondsuit Pe_k.$$

If u is an elementary tensor, one can easily verify that $(u\cdot a_k)\diamondsuit Pe_k = (u\odot a_k e_k)\diamondsuit P$. Hence, by bilinearity, the same is true for the general $u\in\mathcal{K}E$. Therefore we have

$$(u\diamondsuit\omega)\cdot b=\sum_{k=1}^n (u\cdot a_k)\diamondsuit Pe_k=\sum_{k=1}^n (u\odot a_k e_k)\diamondsuit P=(u\odot v)\diamondsuit P=U\diamondsuit P.$$

¿From this, combining the expression of the four-named norm by (10) with Propositions 8 and 6, we see that $\|U\|_4=\|U\diamondsuit P\|_4\le\|u\|\|\omega\|\|b\|=\|u\|\|b\|$. But the C^*-identity, together with Propositions 8 and 5, gives $\|b\|=\|v\|$. Therefore, taking all possible representations of U as single Effros symbols and using (7), we complete the proof in the "column" case. The similar argument, with the obvious modifications, works in the "raw" case. $\triangleright$

The following important observation illustrates both Propositions 26 and 28. Let H be a Hilbert space. Denote by $(H\otimes H)_c$ (respectively, $(H\otimes H)_r$ the algebraic tensor square of H, considered as a quantum subspace of the column Hilbertian $(H\dot{\otimes} H)_c$ (respectively, raw Hilbertian $(H\dot{\otimes}H)_r$).

Proposition 29. *Up to complete isometric isomorphisms, $H_c\otimes_4 H_c = H_c\otimes_h H_c = H_c\otimes_{sp} H_c=(H\otimes H)_c$ and $H_r\otimes_4 H_r = H_r\otimes_h H_r = H_r\otimes_{sp} H_r = (H\otimes H)_r$.*

$\triangleleft$ Because of the analogy between the "column" and the "raw" cases, it is sufficient to restrict ourselves with the first chain of equalities. In this chain, the

first two equalities follow from the mentioned propositions. We proceed to the third equality.

Recall that $H_c = \mathcal{B}(\mathbb{C}, H)$ and consider the linear isomorphism $I : H_c \dot{\otimes} H_c \to (H \otimes H)_c$, coinciding, after the respective identifications, with the identity operator on $H \otimes H$. Since we deal with standard quantum spaces, its amplification I_∞ acts between $\mathcal{K} \dot{\otimes} H_c \dot{\otimes} H_c$ and $\mathcal{K} \dot{\otimes} (H \otimes H)_c$. This is obviously a birestriction of a certain isometric isomorphism. The latter, if we want to be meticulous, acts between $\mathcal{B}(L \dot{\otimes} \mathbb{C} \dot{\otimes} \mathbb{C}, L \dot{\otimes} H \dot{\otimes} H)$ and $\mathcal{B}(L \dot{\otimes} \mathbb{C}, L \dot{\otimes} H \dot{\otimes} H)$ and is generated by the natural identification of $L \dot{\otimes} (\mathbb{C} \dot{\otimes} \mathbb{C})$ with $L \dot{\otimes} \mathbb{C}$. Therefore I_∞ is itself an isometric isomorphism. The rest is clear. ▷

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